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Spectral Theory in Normed Vector Spaces

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Abstract

Eigenvalues and eigenvectors are fundamental to linear operators, with applications across mathematics. While well-understood in finite dimensions, the transition to infinite dimensions poses significant challenges requiring nuanced analysis. The primary objective is to establish a comprehensive framework for the functional calculus of linear operators, starting with finite dimensions and extending to infinite-dimensional normed vector spaces. The Riesz-Dunford functional calculus, representing operator functions as contour integrals involving the resolvent, is the central focus. In finite dimensions, the Riesz-Dunford calculus is readily applicable due to bounded linear operators. However, in infinite dimensions, the presence of unbounded operators necessitates more sophisticated techniques and specialized operator classes. This work lays the groundwork for generalizing the Riesz-Dunford calculus to infinite dimensions, encompassing operational calculus, densely defined operators, and generalized spectral theory. The insights have far-reaching implications, bridging finite and infinite dimensions and advancing operator theory in pure and applied mathematics.

Declaration

I, the undersigned, hereby declare that the work contained in this essay is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Steve Ferace Tanefo Mefeza, 18 May 2024.

Contents

Notations

To ensure clarity and consistency throughout this work, the following chapter will provide definitions and explanations of the key notations and concepts used.

Let X be a normed vector space. In this work, we will use the following notations.

1. $\mathcal{L}(X, Y)$: The set of all linear operators from X into Y.
2. $\mathcal{L}(X)$: The set of all linear operators from X to itself.
3. $B(X, Y)$: The set of all bounded operators from X to Y.
4. $\mathbb{P}[\mathbb{C}]$: The set of all polynomials in an indeterminate (or variable) with coefficients from the set $\mathbb{C}$.
5. $\mathbf{Hol}(X)$: The set of all functions $f \longrightarrow \mathbb{C}$ that are holomorphic (complex differentiable) on X.
6. δ_{kl}: Kronecker delta

1. Introduction

1.1 General Introduction

Spectral theory is a fundamental branch of mathematics that investigates the properties and behavior of operators on normed spaces. It provides powerful tools and techniques for understanding the spectral properties of linear operators, which have wide-ranging applications in various fields, including mathematics, physics, engineering, and data analysis. Central to spectral theory is the concept of functional calculus, which extends the notion of functions to operators [12], enabling us to manipulate and analyze operators as if they were functions of the underlying space.

The study of spectral theory involves exploring the spectrum of an operator, which consists of the set of all possible values that the operator can attain. The spectrum provides valuable information about the behavior and properties of the operator, such as its eigenvalues, eigenvectors, and the nature of its resolvent. Functional calculus plays a pivotal role in spectral theory, as it allows us to define functions of operators. Just as we can apply functions to numbers, functional calculus enables us to apply functions to operators, yielding new operators with specific properties. This extension of functions to operators provides a powerful framework for analyzing and manipulating operators in a manner that aligns with familiar notions from classical analysis. One of the fundamental results in functional calculus is the spectral mapping theorem, which establishes a relationship between the spectrum of an operator and the spectrum of its functional calculus [14]. The theorem states that if a function is applied to an operator, then the spectrum of the resulting operator is precisely the image of the original operator's spectrum under the function. This theorem provides a bridge between the spectral properties of an operator and its functional calculus, allowing us to deduce information about the spectrum of the resulting operator.

In the realm of functional calculus, the Riesz Functional Calculus stands out as a powerful tool for understanding and analyzing operators. In the finite-dimensional case, the Riesz Functional Calculus has been extensively studied and well-established. However, when it comes to infinite-dimensional spaces, new challenges arise, and an extension of the calculus is required. Notable authors and their works have significantly influenced the field. Gelfand and Shilov's introduction of functional calculus for bounded linear operators on Banach spaces laid the foundation for subsequent investigations. Dunford and Schwartz's comprehensive treatment of spectral theory in [10, 11], provided valuable insights into functional calculus and its applications. Kato's "Perturbation Theory for Linear Operators" and Reed and Simon's multi-volume series "Methods of Modern Mathematical Physics" have been influential in the study of functional calculus and spectral theory. In 1991, Rudin, W. in his book "Functional Analysis", [18], provides a rigorous treatment of the subject.

1.2 Objectives

The main goal of this work is to understand and present the **Cauchy-Riesz functional calculus.** Our specific objectives are

1. Provide a thorough examination of the calculus and its applications, in a finite dimensional case.
2. Generalize the finite dimensional case to the infinite dimensional Hilbert space cases.

3. Introduce a functional calculus that encompasses the infinite dimensional case.

1.3 Thesis Organization

This essay is divided in 4 chapters. In Chapter 2, we provide the necessary background on normed spaces, laying the foundation for our subsequent discussions. We review some spectral properties of bounded linear operator. Chapter 3 focuses on the functional calculus of bounded operators. We delve into various functional calculi, including the polynomial functional calculus, power series functional calculi, and the Riesz-Dunford functional calculus. In Chapter 4, we extend the functional calculus to infinite-dimensional spaces, which pose unique challenges compared to finite-dimensional spaces. Finally, chapter 5 concludes the work.

2. Preliminaries

2.1 Introduction

In normed spaces, linear operators represent transformations that preserve the linear structure. Spectral theory allows us to analyze these transformations by examining the spectrum of an operator, which is a set of complex numbers that encode important information about the operator's behavior.

The spectrum of an operator provides insights into its eigenvalues and eigenvectors, which are fundamental concepts in linear algebra. Eigenvalues represent the values at which an operator fails to be invertible, while eigenvectors are the corresponding non-zero vectors that are mapped to scalar multiples of themselves by the operator.

More details about this chapter can be found in the books [3, 9, 10, 13, 15, 16, 17, 21]

2.2 Foundations of Normed Spaces

2.2.1 Definition. [20] A **normed space** is a pair $(V, \|\cdot\|)$ where V is a complex vector space, and $\|\cdot\|$ a norm on V, a function

$$V \ni x \mapsto \|x\| \in [0, \infty)$$

which satisfies the following conditions, for all $x, y \in V$, $\alpha \in \mathbb{C}$

(C1) (positive definiteness) $\|x\| = 0 \Leftrightarrow x = 0$,

(C2) (homogeneity) $\|\alpha x\| = |\alpha| \|x\|$,

(C3) (triangle inequality) $\|x + y\| \leq \|x\| + \|y\|$.

2.2.2 Definition. [14] A **Banach space** is a normed space which is complete in the metric defined by the norm given by:

$$d(x, y) = \|x - y\| \quad \forall x, y \in X \tag{2.2.1}$$

2.2.3 Definition. [20, Theorem 2.3]

(a) An **inner product** on a (complex) vector space V is a mapping $V \times V \to \mathbb{C}$, denoted as $(x, y) \mapsto \langle x, y\rangle$ which satisfies the following conditions, for all $x, y, z \in V$ and $\alpha \in \mathbb{C}$:

   (i) (positive definiteness) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0 \Leftrightarrow x = 0$;
   (ii) (hermitian symmetry) $\langle x, y\rangle = \overline{\langle y, x\rangle}$;
   (iii) (linearity in first variable) $\langle \alpha x + \beta z, y\rangle = \alpha\langle x, y\rangle + \beta\langle z, y\rangle$.

   An **inner product space** is a vector space endowed with an inner product.

   With an inner product on V, we can define:

   - A norm on V given by
$$\|x\| = \sqrt{\langle x, x\rangle}. \tag{2.2.2}$$

- A metric on V given by

$$d(x,y) = \|x-y\| = \sqrt{\langle x-y, x-y\rangle}. \tag{2.2.3}$$

(b) An inner product space which is complete in the norm coming from the inner product is called a **Hilbert space**.

2.2.4 Example. The following spaces are normed and Banach spaces.

(a) $C[a,b]$, the vector space of all continuous functions on the compact $[a,b]$, equipped with the norm defined by:

$$\forall\ 1 \le p < \infty, \quad \|f\|_p = \left(\int_a^b |f(x)|^p dx\right)^{1/p}.$$

By the Minkowski's inequality for functions, one sees that $(C[a,b], \|\cdot\|_p)$ forms a normed space under this norm [7].

(b) The space of continuous functions on a compact interval, denoted by $C[a,b]$, with the sup norm:

$$\|f\| = \sup_{x\in[a,b]} |f(x)|. \tag{2.2.4}$$

2.2.5 Example. The following spaces are inner product spaces.

(a) For $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{C}^n$, the inner product $\langle x, y\rangle$ is defined as:

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i \overline{y_i}. \tag{2.2.5}$$

It can be easily verified that this definition establishes an inner product on $\mathbb{C}^n$.

(b) The equation

$$\langle f, g\rangle = \int_{[0,1]} f(x)\overline{g(x)}\, dx \tag{2.2.6}$$

is easily verified to define an inner product on $C[0,1]$.

(c) The space $L^2([0,1])$ is a Hilbert space because it is a complete inner product space. This means that every Cauchy sequence in $L^2([0,1])$ converges to a limit in $L^2([0,1])$.

2.2.6 Definition. A vector space X is called a **Banach algebra** (with unit) if :

1. X is a Banach space.
2. X is equipped with a multiplication operation $X \times X \longrightarrow X$ that possesses the following properties, for all $x, y, z \in X, c \in \mathbb{C}$:
   (i) $(xy)z = x(yz)$
   (ii) $(x+y)z = xz + yz$
   (iii) $x(y+z) = xy + xz$
   (iv) $c(xy) = (cx)y = x(cy)$

(v) there is a unit element e, such that $ex = xe = x$, for all $x \in X$.

3. $\|e\| = 1$

4. $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in X$.

Therefore, a Banach algebra is a mathematical structure that combines the properties of an algebra (a vector space with multiplication satisfying algebraic rules) and a Banach space, with both structures being compatible.

2.3 Spectral Properties of Bounded Linear Operators

2.3.1 Definition. Linear mapping. Let X and Y two vector spaces over the same scalar field. The map [18] $L : X \to Y$ is said to be linear if

$$L(\alpha x + \beta y) = \alpha L(x) + \beta L(y)$$

for all x and y in X and all scalars α and β.

2.3.2 Definition. A **Bounded linear operator** is a mapping L of a topological vector space X into a topological vector space Y such that $L(M)$ is a bounded subset in Y for any bounded subset M of X.

Every continuous operator $L : X \to Y$ is a bounded operator. If $L : X \to Y$ is a linear operator, then for L to be bounded it is sufficient that there exists a neighborhood $U \subset X$ such that $L(U)$ is bounded in Y, see in [18].

2.3.3 Definition. Adjoint of a linear operator. Let T be a linear operator on a Hilbert space $\mathcal{H}$. The adjoint of T, denoted by T^*, is the unique linear operator on $\mathcal{H}$ that satisfies the following condition for all $x, y \in \mathcal{H}$:

$$\langle Tx, y\rangle = \langle x, T^* y\rangle.$$

2.3.4 Example. Let T be the linear operator on $\mathbb{C}^n$ given by the matrix $T = \begin{pmatrix} 1+i & 2-4i \\ 3 & 4+7i \end{pmatrix}$. Then the adjoint of T is the linear operator on $\mathbb{C}^n$ given by the matrix $\begin{pmatrix} 1-i & 3 \\ 2+4i & 4-7i \end{pmatrix}$. It's simply the conjugate-transpose of T.

2.3.5 Definition. A **bounded normal operator** is a bounded linear operator on a Hilbert space that commutes with its adjoint. In other words, if T is a bounded linear operator on a Hilbert space $\mathcal{H}$, then T is normal if and only if $TT^* = T^*T$.

2.3.6 Proposition. If X and Y are normed vector spaces, then L is bounded if and only if there exists some $M > 0$ such that for all $x \in X$,

$$\|Lx\|_Y \leq M\|x\|_X. \tag{2.3.1}$$

2.3.7 Definition. The **norm** of an operator $L : X \to Y$, denoted $\|L\|$, is the smallest M such that inequality (2.3.1) is satisfied, in other words,

$$\|L\| = \sup_{x \neq 0} \frac{\|Lx\|_Y}{\|x\|_X}.$$

Let X be a finite dimensional normed space, and $T : X \to X$ a linear bounded operator. T can be represented by matrices, which depend on the choice of bases for X. Actually, spectral theory of T is mostly matrix eigenvalue theory.

2.3.8 Definition. The **spectrum** of a bounded linear operator T on a normed space X, denoted by $\sigma(T)$, is the set of all complex numbers λ such that $T - \lambda I$ is not invertible, meaning that $T - \lambda I$ does not have a bounded inverse.

1. **Significance**
   - The spectrum provides information about the behavior of the operator.
   - The spectrum can be used to determine the eigenvalues and eigenvectors of the operator.
   - The spectrum can be used to analyze the stability and convergence of the operator.
2. **Types of spectra**
   - **Point spectrum** (eigenvalues): The set of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not injective.
   - **Continuous spectrum**: The set of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is injective but not surjective.
   - **Residual spectrum**: The set of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is surjective but not injective.

2.3.9 Theorem. (Gelfand) *[17, Theorem 1.2.5, p. 9]. The spectrum of a linear operator, T, is non-empty, $\sigma(T) \neq \emptyset$.*

2.3.10 Definition. Spectral radius. The spectral radius of an operator T is defined to be

$$r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$$

2.3.11 Theorem. *Let X be a Banach algebra, T, a linear operator on X then :*

$$r(T) = \inf_{n \geq 1} \|T^n\|^{1/n} = \lim_{n \to \infty} \|T^n\|^{1/n}. \tag{2.3.2}$$

For a proof, see [17, Theorem 1.2.7, p. 10]

2.3.12 Definition. The **resolvent** of an operator T is the bounded linear operator $R_\lambda(T) = (T - \lambda I)^{-1}$, where $\lambda \in \mathbb{C}$ is not in the spectrum of T. The set, $\rho(T)$, of all complex numbers λ such that $T - \lambda I$ is invertible is called the **resolvent set** of T.

The resolvent set satisfies the following properties:

- The resolvent is a bounded linear operator (by definition of the spectrum).
- The resolvent is analytic in the resolvent set (see Chap 3, Section 1).
- The spectrum of T is the complement of the resolvent set (by definition).

2.3.13 Theorem. Representation Theorem (Resolvent). *For a bounded linear operator T on a Banach space X, and $\lambda_0 \in \rho(T)$, the resolvent $R_\lambda(T)$ has the representation*

$$R_\lambda(T) = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k R_{\lambda_0}^{k+1}, \tag{2.3.3}$$

the series being absolutely convergent for every λ in the open disk (in the complex plane)

$$|\lambda - \lambda_0| < \frac{1}{\|R_{\lambda_0}\|}.$$

Proof. Let $\lambda_0 \in \rho(T)$, for all $\lambda \in \mathbb{C}$, we have

$$T_\lambda \equiv T - \lambda I = T - \lambda_0 I - (\lambda - \lambda_0)I = (T - \lambda_0 I)[I - (\lambda - \lambda_0)(T - \lambda_0 I)^{-1}]$$

By setting $V_\lambda = I - (\lambda - \lambda_0)R_{\lambda_0}$, we obtain

$$T_\lambda = T_{\lambda_0} V_\lambda. \tag{2.3.4}$$

Since $\lambda_0 \in \rho(T)$ and T is bounded, $R_{\lambda_0} = T_{\lambda_0}^{-1}$ is bounded. Moreover, for all λ with $\|(\lambda - \lambda_0)R_{\lambda_0}\| < 1$, the operator V_λ has an inverse (see [14, Theorem 7.3-1, P. 375]), given by

$$V_\lambda^{-1} = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k R_{\lambda_0}^k.$$

For all λ satisfying

$$|\lambda - \lambda_0| < \frac{1}{\|R_{\lambda_0}\|}$$

The operator T_λ in (2.3.4) has an inverse

$$R_\lambda(T) = T_\lambda^{-1} = (T_{\lambda_0} V_\lambda)^{-1} = V_\lambda^{-1} R_{\lambda_0} = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k R_{\lambda_0}^{k+1}. \tag{2.3.5}$$

Hence, the theorem is proved.

□

2.3.14 Theorem. First resolvent equation, commutativity. *Let X be a complex Banach space, $T \in B(X, X)$, set of bounded operator from X to X, and $\lambda_1, \lambda_2 \in \rho(T)$. Then:*

(i) The resolvent R_λ of T satisfies the Hilbert relation or resolvent equation

$$R_{\lambda_1} - R_{\lambda_2} = (\lambda_1 - \lambda_2) R_{\lambda_1} R_{\lambda_2}. \tag{2.3.6}$$

(ii) R_λ commutes with any $S \in B(X, X)$ which commutes with T.

(iii) We have for all $\lambda_1, \lambda_2 \in \rho(T)$,

$$R_{\lambda_1} R_{\lambda_2} = R_{\lambda_2} R_{\lambda_1}. \tag{2.3.7}$$

Proof. (i) Let $\lambda_1, \lambda_2 \in \rho(T)$, we have $I = R_{\lambda_1} T_{\lambda_1}$ and $I = T_{\lambda_2} R_{\lambda_2}$, where $T_\lambda = T - \lambda I$ then:

$$\begin{aligned} R_{\lambda_1} - R_{\lambda_2} &= R_{\lambda_1} I - I R_{\lambda_2} \\ &= R_{\lambda_1}(T - \lambda_2 I) R_{\lambda_2} - R_{\lambda_1}(T - \lambda_1 I) R_{\lambda_2} \\ &= R_{\lambda_1} \left[T - \lambda_2 I - (T - \lambda_1 I)\right] R_{\lambda_2} \\ &= R_{\lambda_1}(\lambda_1 - \lambda_2) R_{\lambda_2}. \end{aligned}$$

(ii) Let $S \in B(X, X)$ with $ST = TS$. Then, one has:

$$S(T - \lambda I) = ST - S\lambda I = TS - \lambda IS = (T - \lambda I)S.$$

That is,

$$ST_\lambda = T_\lambda S$$

Using $I = T_\lambda R_\lambda = R_\lambda T_\lambda$, we thus obtain:

$$R_\lambda S = R_\lambda S T_\lambda R_\lambda = \underbrace{(R_\lambda T_\lambda)}_{I} S R_\lambda = S R_\lambda.$$

(iii) Let $\lambda_1, \lambda_2 \in \rho(T)$. Thus,

$$\begin{cases} TR_{\lambda_1} = T(T-\lambda_1 I)^{-1} = \left[(T-\lambda_1 I)\, T^{-1}\right]^{-1} = \left(I - \lambda_1 T^{-1}\right)^{-1} \\ R_{\lambda_1} T = (T-\lambda_1 I)^{-1} T = \left[T^{-1}\,(T-\lambda_1 I)\right]^{-1} = \left(I - \lambda_1 T^{-1}\right)^{-1}, \end{cases}$$

and hence

$$R_{\lambda_1} T = TR_{\lambda_1}. \tag{2.3.8}$$

As R_{λ_1} commute with T, then (according to (ii)), we have

$$R_{\lambda_1} R_{\lambda_2} = R_{\lambda_2} R_{\lambda_1}.$$

□

2.3.15 Lemma. Holomorphy of R_λ. [19, Proposition 5.2.2, p. 203] The resolvent $R_\lambda(T)$ of a bounded linear operator $T : X \longrightarrow Y$ on a complex Banach space X is holomorphic at every point λ of the resolvent set $\rho(T)$ of T. Hence it is locally holomorphic on $\rho(T)$.

Proof. In Theorem 2.3.13 we have shown that R_λ can be represented by a convergent power series in a neighborhood of any point $\lambda_0 \in \sigma(T)$. Hence, the function R_λ is analytic on $\sigma(T)$, thereby continuous on $\sigma(T)$.

From the resolvent equation (2.3.6), we have

$$\frac{R_{\lambda_1} - R_{\lambda_2}}{\lambda_1 - \lambda_2} = R_{\lambda_1} R_{\lambda_2} \quad \forall \lambda_1, \lambda_2 \in \rho(T), \tag{2.3.9}$$

Taking the limit when $\lambda_2 \longrightarrow \lambda_1$ in both side of (2.3.9), and using the fact that R_λ is continuous in λ_1, we have:

$$R'_{\lambda_1} = R^2_{\lambda_1}.$$

Thus, R_λ is differentiable at each point $\lambda \in \rho(T)$, that is, R_λ is **holomorphic in** $\rho(T)$ and $R'_\lambda = R^2_\lambda$.

□

2.3.16 Definition. (Eigenvalues and Eigenvectors)

An **eigenvalu**e of an operator T is a complex number λ such that there exists a non-zero vector x such that $Tx = \lambda x$.

An **eigenvector** of an operator T is a non-zero vector x such that $Tx = \lambda x$ for some eigenvalue λ.

2.3.17 Example. $x_1 = \binom{1}{2}$ and $x_2 = \binom{1}{-2}$ are the eigenvectors of, $T = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$ corresponding to the eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -1$.

3. Functional Calculus of Bounded Operator

Functional Calculus is a mathematical framework that extends the concept of functions to operators on a given vector space. It provides a way to evaluate functions of operators, such as exponentials, logarithms, polynomials, and more. Functional calculus is particularly useful in the study of linear operators.

In functional calculus, the idea is to define a map that associates a function with an operator, allowing us to apply the function to the operator and obtain a new operator. If f is a function, say a numerical function of a real number, and T is an operator, there is no particular reason why the expression $f(T)$ should make sense.

3.1 Polynomial Functional Calculus

Let X be a Banach space over $\mathbb{C}$, and consider a bounded operator T on X. Then, for any polynomial

$$p(z) = \sum_{i=0}^{m} a_i z^i, \tag{3.1.1}$$

one can simply substitute T for z and define

$$p(T) := \sum_{i=0}^{m} a_i T^i, \tag{3.1.2}$$

where $T^0 = I$.

The map

$$\Phi : \mathbb{P}[\mathbb{C}] \to \mathcal{L}(X), \quad p \longmapsto \Phi(p) := p(T)$$

is a unital algebra homomorphism. That is, Φ is a **representation** of the unital algebra $\mathbb{C}(z)$ on the vector space of bounded operators on X.

Note that Φ is the **polynomial functional calculus.**

3.2 Power Series Functional Calculi

So far, we have not taken into account the boundedness of the operator T. Let us incorporate the boundedness condition and expand the polynomial functional calculus slightly. Assume f is an entire function on $\mathbb{C}$ ($f \in \textbf{Hol}(\mathbb{C})$) with a power series representation,

$$f = \sum_{i=0}^{\infty} a_i \mathbf{z}^i, \tag{3.2.1}$$

then one has

$$f(T) := \Phi(f) := \sum_{i=0}^{\infty} a_i T^i \in \mathcal{L}(X). \tag{3.2.2}$$

Given the fact that the MacLaurin series of an entire function converges everywhere, the series in (3.2.2) converges, therefore, $f(T)$ makes sense.

Indeed, $\sum\limits_{i=0}^{\infty} a_i T^i$ converges, because the absolute series

$$\sum_{i=0}^{\infty} |a_i| \|T^i\| \leq \sum_{i=0}^{\infty} |a_i| \|T\|^i \tag{3.2.3}$$

converges. Furthermore, X is complete, then the series $\sum\limits_{i=1}^{\infty} a_i T^i$ converges

When $f(z) = \sum_{i=0}^{n} a_i z^i$ is a polynomial, we recover the **polynomial functional calculus**.

3.2.1 Example. Exponential of a matrix T **.**

$f : z \longmapsto e^z$ is holomorphic on $\mathbf{C}$. Replacing z by the operator T in the power series representation of f, we get

$$f(T) = e^T = I + T + \frac{T^2}{2!} + \frac{T^3}{3!} + \cdots .$$

If f is the function $z \longmapsto \ln(z+1)$, then the MacLaurin series of f is given by $f(z) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} z^n$. The radius of convergence of this series is 1, so the formulae in (3.2.2) fails when $\|T\| \geq 1$ (operator norm). We then need a general functional calculus.

3.3 Riesz–Dunford Functional Calculus

In mathematics, holomorphic functional calculus is functional calculus with holomorphic functions. More precisely, the functional calculus defines a continuous algebra homomorphism from the holomorphic functions on a neighborhood of the spectrum of T to the bounded operators. The functional calculus of Riesz–Dunford $(\mathbf{Hol}(\sigma(T)))$, that is, all functions holomorphic on the spectrum $\sigma(T)$ of the operator T.

Recall Cauchy Theory [4, Example 3.7.4, p. 200]. For $m \in \mathbb{Z}$ and $a \in \mathbb{C}$, consider the function of the complex variable z defined by $f(z) = \frac{1}{(z-a)^m}$. This function is defined and holomorphic on the connected (but not simply connected) domain $D = \mathbb{C} \setminus \{a\}$. Consider also a circuit Γ consisting of a circle of radius r centered at the point a. Then,

$$\oint_{\Gamma} \frac{1}{(z-a)^m} dz = 2\pi i \delta_{m,1} = \begin{cases} 0 & \text{if } m \neq 1, \\ 2\pi i & \text{if } m = 1. \end{cases}$$

3.3.1 Lemma. Let γ be a positively oriented simply closed and piecewise smooth contour lying in the region $|\lambda| > spr(T)$. The following formula holds:

$$\frac{1}{2\pi i} \int_{\gamma} \lambda^m R(\lambda, T) d\lambda = T^m. \tag{3.3.1}$$

Proof. [6, section 2.3.1, p. 63] For all $|\lambda| > spr(T)$, the resolvent operator$R(\lambda, T)$ is given by

$$R(\lambda,T) = (T-\lambda I)^{-1} = \frac{1}{\lambda}\left(\frac{T}{\lambda} - I\right)^{-1} = \frac{1}{\lambda}\sum_{n\geq 0}\left(\frac{T}{\lambda}\right)^{n} = \sum_{n\geq 0}\frac{T^n}{\lambda^{n+1}},$$

and it converges. Thus, we obtaint

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \lambda^m R(\lambda,T)\,d\lambda &= \frac{1}{2\pi i}\int_\gamma \lambda^m \sum_{n\geq 0}\frac{T^n}{\lambda^{n+1}}\,d\lambda \\ &= \frac{1}{2\pi i}\sum_{n\geq 0} T^n \int_\gamma \frac{1}{\lambda^{n+1-m}}\,d\lambda \\ &= \frac{1}{2\pi i}\sum_{n\geq 0} T^n (2\pi i \delta_{n,m}) \\ &= T^m.\end{aligned}$$

□

3.3.2 Cauchy-Riesz Integral. Here, the idea is to extend the Cauchy integral formula from classical function theory to functions taking values in the Banach space $\mathcal{L}(X)$.

Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function on $\sigma(T)$ (a neighborhood of the spectrum of T). Let $\gamma = \{\gamma_1, \gamma_2, ..., \gamma_m\}$ be a suitable collection of positively oriented simply closed contours of integration which consist of regular points and lie inside the holomorphic region of f, where the eigenvalues are inside the contours. Then one defines the new operator

$$f(T) := \frac{1}{2\pi i}\int_\gamma f(\lambda)R(\lambda,T)d\lambda. \tag{3.3.2}$$

It follows from Lemma 2.3.15 and from the Cauchy integral theorem (see [4, Theorem 3.8.1, p 208]), that $f(T)$ depends only on the function f, and not on the contour γ. Therefore, the map

$$\Phi : \mathbf{Hol}(\sigma(T)) \to \mathcal{L}(X), \quad \Phi(f) := f(T) \tag{3.3.3}$$

is well-defined. It is called the **Riesz-Dunford Functional Calculus**.

3.3.3 Theorem. *Suppose T is a bounded linear operator on a Banach space X. The mapping Φ defined by equation (3.3.3) is a homomorphism, with $\Phi(\mathbf{z}) = T$, where $\mathbf{z}$ represents the function $\mathbf{z}(z) = z$. Additionally, Φ is continuous under local uniform convergence. In other words, if $(f_n)_{n\in\mathbb{N}}$ is a sequence in Hol$(\sigma(T))$ that converges to f locally uniformly on $\sigma(T)$, then $\Phi(f_n)$ converges to $\Phi(f)$ in the operator norm.*

Proof. 1. Let's show that Φ is a homomorphism. See [12, section 1, p6] for more details.

Let $f_1, f_2 \in \mathbf{Hol}(\sigma)$. Now,

- $\Phi(f_1 + f_2) = \Phi(f_1) + \Phi(f_2)$ by definition of Φ
- $\Phi(f_1 f_2) = \Phi(f_1)\Phi(f_2)$. Indeed, we choose γ_1 and γ_2 satisfying the appropriate assumptions, and such that γ_1 lies in the inside of γ_2, as shown by Figure 3.1.

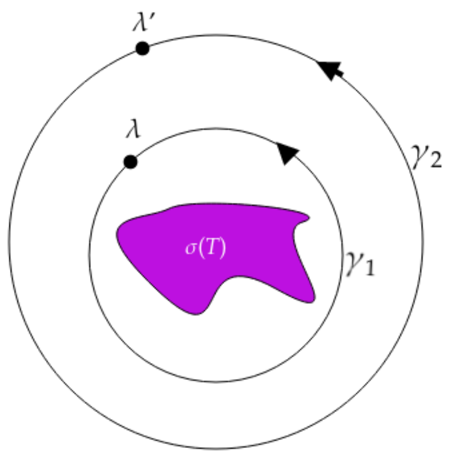

Figure 3.1: contours of integration γ_1 and γ_2.

Applying First Resolvent Formula and Fubini's Theorem, we have

$$f_1(T)f_2(T) = \left(\frac{1}{2\pi i}\int_{\gamma_1}\frac{f_1(\lambda)}{\lambda - T}\,d\lambda\right)\left(\frac{1}{2\pi i}\int_{\gamma_2}\frac{f_2(\lambda')}{\lambda' - T}\,d\lambda'\right)$$

$$= \frac{1}{(2\pi i)^2}\int_{\gamma_1}\int_{\gamma_2}\frac{f_1(\lambda)f_2(\lambda')}{(\lambda - T)(\lambda' - T)}\,d\lambda'\,d\lambda$$

$$= \frac{1}{(2\pi i)^2}\int_{\gamma_1}\int_{\gamma_2} f_1(\lambda)f_2(\lambda')\left(\frac{(\lambda - T)^{-1} - (\lambda' - T)^{-1}}{\lambda' - \lambda}\right)d\lambda'\,d\lambda$$

$$= \frac{1}{(2\pi i)^2}\left\{\left(\int_{\gamma_1}\frac{f_1(\lambda)}{\lambda - T}\left[\int_{\gamma_2}\frac{f_2(\lambda')}{\lambda' - \lambda}\,d\lambda'\right]d\lambda\right) - \left(\int_{\gamma_2}\frac{f_2(\lambda')}{\lambda' - T}\left[\int_{\gamma_1}\frac{f_1(\lambda)}{\lambda' - \lambda}\,d\lambda\right]d\lambda'\right)\right\}$$

Since $\lambda' \in \gamma_2$ lies outside γ_1 (see Figure 3.1), we obtain

$$\int_{\gamma_1}\frac{f_1(\lambda)}{\lambda' - \lambda}\,d\lambda = 0. \tag{3.3.4}$$

Using (3.3.4) and applying Cauchy's Theorem we obtain

$$f_1(T)f_2(T) = \frac{1}{(2\pi i)^2}\int_{\gamma_1}\frac{f_1(\lambda)}{\lambda - T}\underbrace{\int_{\gamma_2}\frac{f_2(\lambda')}{\lambda' - \lambda}\,d\lambda'}_{2\pi i f_2(\lambda)}\,d\lambda$$

$$= \frac{1}{2\pi i}\int_{\gamma_1}\frac{f_1(\lambda)f_2(\lambda)}{\lambda - T}d\lambda$$

Therefore,

$$\Phi(f_1)\Phi(f_2) = \frac{1}{2\pi i}\int_{\gamma_1}(f_1f_2)(\lambda)R(\lambda, T)d\lambda = \Phi(f_1f_2). \tag{3.3.5}$$

2. Let's show that Φ is Continuous (with respect to compact convergence). Suppose $(f_n)_{n\in\mathbb{N}} \subseteq Hol(\sigma(T))$ such that $f_n \to f$ uniformly, let's show that $\Phi(f_n) \to \Phi(f)$ in the uniform topology of operator. Now,

$$\|\Phi(f_n) - \Phi(f)\| = \left\| \frac{1}{2\pi i} \int_\gamma (f_n(\lambda) - f(\lambda))\, R(\lambda, T) d\lambda \right\|. \tag{3.3.6}$$

Let us consider the parameterization of the contour γ given by $\gamma = \{z(t), t \in [0,1]\}$. The equation (3.3.6) becomes

$$\begin{aligned}\|\Phi(f_n) - \Phi(f)\| &= \left\| \frac{1}{2\pi i} \int_0^1 (f_n - f)\,(z(t)) R(z(t), T) z'(t)\, dt \right\| \\ &\leq \frac{1}{2\pi} \int_0^1 |(f_n - f)(z(t))|\, \|R(z(t), T)\|\, |z'(t)|\, dt \\ &\leq \frac{1}{2\pi} \|f_n - f\|_\infty \int_0^1 \|R(z(t), T)\|\, |z'(t)|\, dt.\end{aligned}$$

Since $f_n \xrightarrow{u} f$, the norm $\|f_n - f\|_\infty$ tends to 0. Hence,

$$\|\Phi(f_n) - \Phi(f)\| \to 0.$$

Therefore, the proof is done.

□

3.3.4 Theorem. *[11, Theorem 5, p. 558] If f has the power series expansion $f(\lambda) = \sum_{i=0}^\infty a_i \lambda^i$, valid in a neighborhood of $\sigma(T)$, then $f(T) = \sum_{i=0}^\infty a_i T^i$.*

Proof. The result comes from Lemma 3.3.2. Let $f(\lambda) = \sum_{i=0}^\infty a_i \lambda^i$, such that it converges for $|\lambda| < r(T) + \epsilon, \epsilon > 0$, it follows that:

$$f(T) = \frac{1}{2\pi i} \int_\gamma f(\lambda) R(\lambda, T) d\lambda = \frac{1}{2\pi i} \int_\gamma \sum_{i=0}^\infty a_i \lambda^i R(\lambda, T) d\lambda = \frac{1}{2\pi i} \sum_{i=0}^\infty a_i \int_\gamma \lambda^i R(\lambda, T) d\lambda = \sum_{i=0}^\infty a_i T^i.$$

□

We then recover the **power series functional calculus**.

3.4 Spectral Mapping Theorem

The spectrum of $f(T)$ is derived from the one of T by a simple formula.

3.4.1 Theorem. *Let T be a bounded operator on a Banach space X let U open, $\sigma(T) \subseteq U$ and let $f \in \mathbf{Hol}(U)$. Then the following equality hols.*

$$f(\sigma(T)) = \sigma(f(T)).$$

Proof. $\subseteq$) For the inclusion $f(\sigma(T)) \subset \sigma(f(T))$, we proceed by contradiction. Suppose $\exists \mu \in \mathbb{C}$ such that $\mu \in f(\sigma(T))$ and $\mu \notin \sigma(f(T))$. That implies

$$\exists \lambda_0 \in \sigma(T) \text{ such that } \mu = f(\lambda_0); \quad f(\lambda_0) \notin \sigma(f(T))$$
$$\exists \lambda_0 \in \sigma(T) \text{ such that } \mu = f(\lambda_0); \quad f(T) - f(\lambda_0)I \text{ is invertible}$$

Define a function g on U by,

$$g(\lambda) = \begin{cases} \frac{f(\lambda)-f(\lambda_0)}{\lambda-\lambda_0} & \text{if } \lambda \neq \lambda_0, \\ \\ f'(\lambda_0) & \text{if } \lambda = \lambda_0. \end{cases}$$

This function g is holomorphic on the punctured neighbourhood $U \backslash \{\lambda_0\}$ and it is continuous at λ_0.

The function $\lambda \longmapsto g(\lambda)(\lambda - \lambda_0)$ is holomorphic on U and satisfies

$$g(\lambda)(\lambda - \lambda_0) = f(\lambda) - f(\lambda_0) \quad \forall \lambda \in U,$$

that implies

$$g(T)(T - \lambda_0) = f(T) - f(\lambda_0)I.$$

Since $f(T) - f(\lambda_0)I$ is invertible, we have:

$$g(T)(T - \lambda_0)(f(T) - f(\lambda_0))^{-1} = I.$$

That means $T - \lambda_0$ is invertible, which contradicts $\lambda_0 \in \sigma(T)$. Therefore, $f(\sigma(T)) \subset \sigma(f(T))$.

$\supseteq$) Now, let's prove that $\sigma(f(T)) \subseteq f(\sigma(T))$. We proceed by contraposition. Let $\mu \in \mathbb{C}$, we want to show that $\mu \notin f(\sigma(T))$ implies $\mu \notin \sigma(f(T))$.

For $\mu \notin f(\sigma(T))$, the function $g : \lambda \longmapsto f(\lambda) - \mu$ is both holomorphic in a neighbourhood of $\sigma(T)$ and is nonzero on $\sigma(T)$. Hence, $g(T) = f(T) - \mu I$ is invertible, meaning that $\mu \notin \sigma(f(T))$. $\square$

3.4.2 Example. Let consider $T = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$, $f(z) = 2z^3$, an entire function. The Spectral Mapping Theorem gives us:

$$\sigma(f(A)) = f(\sigma(A)) = \left\{2(-1)^3, 2(3)^3\right\} = \{-2, 54\}.$$

3.5 Riesz Eigenprojection

An **eigenprojection**, also known as a spectral projection, is a specific type of projection operator associated with an eigenvalue and its corresponding eigenspace. It is a linear transformation that projects any vector in the original vector space onto the eigenspace associated with that eigenvalue.

The **Riesz eigenprojection**, also known as the Riesz spectral projection, is a specific type of eigenprojection associated with a closed subset of the spectrum of a bounded linear operator on a complex Hilbert space. It is a generalization of the concept of an eigenprojection for individual eigenvalues to sets of eigenvalues.

3.5.1 Definition. Spectral set. Let X be a Hilbert space, $T \in \mathcal{L}(X)$. A subset $\sigma \subset \mathbb{C}$ is called a spectral set of T if it satisfies the following properties:

- **Open** and **Closed**: The set σ is both open and closed in the relative topology of the spectrum $\sigma(T)$. This means that every point in σ has a neighborhood that is entirely contained in σ, and every point outside σ has a neighborhood that is entirely contained in the complement of σ.
- **Non-Empty**: The set σ is non-empty, meaning that it contains at least one complex number.

To each spectral set σ, there corresponds a unique projection operator P_σ called the spectral projection associated with σ.

3.5.2 Definition. Spectral projection. The Riesz spectral projection P_σ associated with σ is defined as:

$$P_\sigma = \frac{1}{2\pi i}\int_\Gamma R(\lambda, T)d\lambda$$

where Γ is a simple closed curve in the complex plane that encloses the set σ, and is such that no other eigenvalue lies in its inside or on it.

This projection operator projects the Hilbert space H onto the closed subspace spanned by the eigenvectors of, T whose eigenvalues lie in σ.

3.5.3 Proposition. The Riesz eigenprojection P_σ is a projection operator, that satisfies the following properties:

(P1) **Idempotence** : $P_\sigma^2 = P_\sigma$

(P2) **Orthogonality**: $P_\sigma P_\mu = 0$ for $\sigma \cap \mu = \emptyset$

(P3) **Completeness** : $P_\sigma + P_\mu = P_{\sigma\cup\mu}$ for $\sigma \cap \mu = \emptyset$

(P4) **Spectral Relationship** : $TP_\sigma = P_\sigma T$

Proof. (P1) In Theorem 3.3.4, we have shown (equation (3.3.5)) that :

$$\Phi(f_1)\Phi(f_2) = \Phi(f_1 f_2) \quad \forall f_1, f_2 \in \mathbf{Hol}(\sigma(T))$$

For $f_1 = f_2 = 1$, we have $\Phi^2(1) = \Phi(1)$ that is $P_\sigma^2 = P_\sigma$. Hence, (P1) is proved.

(P2) Let σ, μ spectral sets such that $\sigma \cap \mu = \emptyset$. Thus, we have:

$$P_\sigma P_\mu = \left(\frac{1}{2\pi i}\int_{\Gamma_1} R(\lambda_1, T)\, d\lambda_1\right)\left(\frac{1}{2\pi i}\int_{\Gamma_2} R(\lambda_2, T)\, d\lambda_2\right)$$

$$= \frac{1}{(2\pi i)^2}\int_{\Gamma_1}\int_{\Gamma_2} R(\lambda_1, T)R(\lambda_2, T)\, d\lambda_1\, d\lambda_2$$

$$= \frac{1}{(2\pi i)^2}\int_{\Gamma_1}\int_{\Gamma_2} \frac{R(\lambda_1, T) - R(\lambda_2, T)}{\lambda_1 - \lambda_2}\, d\lambda_2\, d\lambda_1$$

$$= \frac{1}{(2\pi i)^2}\left\{\int_{\Gamma_1} R(\lambda_1, T)\left(\int_{\Gamma_2}\frac{1}{\lambda_1 - \lambda_2}\, d\lambda_2\right) d\lambda_1 - \int_{\Gamma_2} R(\lambda_2, T)\left(\int_{\Gamma_1}\frac{1}{\lambda_1 - \lambda_2}\, d\lambda_1\right) d\lambda_2\right\}.$$

Since $\sigma \cap \mu = \emptyset$ then λ_1 lies outside Γ_2 and λ_2 lies outside Γ_1. Thus,

$$\int_{\Gamma_2} \frac{1}{\lambda_1 - \lambda_2}\, d\lambda_2 = \int_{\Gamma_1} \frac{1}{\lambda_1 - \lambda_2}\, d\lambda_1 = 0.$$

Thereby, $P_\sigma P_\mu = 0$.

(P3) Suppose σ, μ are two disjoint spectral sets, then let Γ_1 and Γ_2 two disjoint contours that enclose σ and μ respectively. We have:

$$P_{\sigma\cup\mu} = \frac{1}{2\pi i}\int_{\Gamma_1\cup\Gamma_2} R(\lambda, T)\, d\lambda = \frac{1}{2\pi i}\left(\int_{\Gamma_1} R(\lambda, T)\, d\lambda + \int_{\Gamma_2} R(\lambda, T)\, d\lambda\right) = P_\sigma + P_\mu.$$

(P4) One has:

$$TP_\sigma = T\int_\Gamma R(\lambda, T)\, d\lambda = \int_\Gamma TR(\lambda, T)\, d\lambda. \tag{3.5.1}$$

We showed in Theorem 2.3.14 (equation (2.3.8)) that $R(\lambda, T)$ commute with T, i.e. $TR(\lambda, T) = R(\lambda, T)T$. Equation (3.5.1) yields then to

$$TP_\sigma = \int_\Gamma R(\lambda, T)T\, d\lambda = \int_\Gamma R(\lambda, T)\, d\lambda T = P_\sigma T.$$

Hence $(P4)$ is proved.

□

3.5.4 Lemma. The function $\sigma \longmapsto P_\sigma$ is a homomorphic map of the Boolean algebra of spectral sets onto a Boolean algebra of projection operators in X and that furthermore this homomorphic takes the unit $\sigma(T)$ in the algebra of spectral sets into the unit I of the algebra of projections.

Proof. See [11, corollary 21, p. 575] □

The previous lemma introduces the notion of spectral measure on X.

3.5.5 Spectral Measure. The collection of all spectral projections associated with spectral sets forms a spectral measure, which is a map from the set of all spectral sets to the set of projection operators on H. The spectral measure provides a way to represent the "distribution" of the spectrum of an operator across different subspaces of the Hilbert space.

3.5.6 Spectral Decomposition. The spectral theorem states that every bounded linear operator on a complex Hilbert space can be decomposed into a direct sum of multiplication operators on function spaces. This decomposition is based on the spectral measure and allows us to understand the structure of the operator and its spectrum in a more comprehensive way.

Let's denote by $X_\sigma = P_\sigma X$, the range of the operator P_σ, $T_\sigma = T|_{X_\sigma}$, the restriction of the operator T to the space X_σ , thus if $\sigma_1, ..., \sigma_n$ are disjoint spectral sets of T such that their union is the whole set $\sigma(T)$, then

$$X = \bigoplus_{i=1}^{n} X_{\sigma_i} \quad \text{and} \quad T = \bigoplus_{i=1}^{n} T_{\sigma_i}. \tag{3.5.2}$$

Hence, to study the structure of T, it is sufficient to study the structure of T_{σ_i}.

3.6 Eigennilpotent Operator

3.6.1 Definition. Nilpotent Operator. [5] An operator N is said to be **nilpotent** if there exists an integer $n \geq 2$ such that

$$N^n = 0, \; N^{n-1} \neq 0. \tag{3.6.1}$$

3.6.2 Proposition. Let $N \in \mathcal{L}(X)$, if N is nilpotent, then one has $N^{dimX} = 0$.

Proof. see [5, chap 8, p. 248]. □

3.6.3 Proposition. (P1) Let $N \in \mathcal{L}(X)$, if N is nilpotent, then the spectrum of N is reduced to 0, $\sigma(N) = \{0\}$.

(P2) Assume X is a complex vector space, $N \in \mathcal{L}(X)$, and the only eigenvalue of N is 0. Then N is nilpotent.

Proof. (P1) Suppose N is nilpotent. Then

$$\exists n \geq 1 \text{ s.t. } N^n = 0, \; N^{n-1} \neq 0$$

Let $\lambda \in \sigma(N)$. Thus,

$$\exists 0 \neq x \in X \text{ such that } Nx = \lambda x$$

Applying N^{n-1}, we get

$$\exists 0 \neq x \in X \text{ such that } N^n x = \lambda N^{n-1} x = \lambda N^{n-2}(Nx) = \lambda^2 N^{n-2} x = \cdots = \lambda^n x$$

Since N^n is the null operator, we have that

$$\lambda^n x = 0 \quad \text{where } x \neq 0$$

It follows that

$$\lambda = 0.$$

(P2) Since X is a complex vector space, the characteristic polynomial (p) of N is split/factorized. Since 0 is the only eigenvalue of N, we have $p(x) = x^n$. According to Cayley-Hamilton theorem,

$$p(N) = 0. \tag{3.6.2}$$

Which is equivalent to,

$$N^n = 0, \tag{3.6.3}$$

i.e. N is nilpotent.

□

Consider the family of operators $\{D_i\}_{1 \leq i \leq n}$ defined by

$$D_i = (T - \lambda_i) P_{\sigma_i} \quad i = 1, 2, \cdots, n, \tag{3.6.4}$$

where T and P_{σ_i} are defined in Section 3.5.

3.6.4 Remark. For all i, $\sigma(D_i) = \{0\}$.

Indeed, let's write D_i as a function of T,

$$D_i = f_i(T); \quad f_i(\lambda) = (\lambda - \lambda_i)l_i(\lambda) \quad \text{and} \quad l_i(\lambda_j) = \delta_{ij} \tag{3.6.5}$$

From the Spectral Mapping Theorem, we have:

$$\sigma(D_i) = \sigma(f_i(T)) = f_i(\sigma(T)) = \{f_i(\lambda_j),\ \lambda_j \in \sigma(T)\} = \{0\}. \tag{3.6.6}$$

A direct implication of this remark is that $\{D_i\}_i$ is a family of nilpotent operators, that is there exists an integer $n_i \geq 1$ such that:

$$D_i^{n_i} = 0, \qquad D_i^{n_i - 1} \neq 0. \tag{3.6.7}$$

For all i, D_i is the eigennilpotent operator associated to the eigenvalue λ_i.

3.6.5 Definition. Index. We call index of the eigenvalue λ_i, the integer $n_i \geq 1$ such that equation (3.6.7) holds.

We note it $n_i = \nu(\lambda_i)$.

4. Functional Calculus in Infinite-Dimension

4.1 Introduction

In the previous chapter, we established a comprehensive framework for functional calculus in finite-dimensional spaces. The Riesz-Dunford functional calculus, in particular, provided a powerful and elegant tool for representing operator functions as contour integrals involving the resolvent operator. This approach was made possible by the fundamental property that all linear operators on finite-dimensional spaces are bounded.

However, the transition from finite to infinite dimensions introduces significant challenges. In infinite-dimensional Banach spaces and Hilbert spaces, the class of bounded linear operators is only a small subset of the full collection of linear operators. Many important operators, such as differential operators and unbounded integral operators, lie outside the realm of bounded operators, and their functional calculus requires a more delicate treatment.

In this chapter, we aim to extend the functional calculus framework developed in the previous chapter to the infinite-dimensional setting. We will introduce a specialized class of operators, such as densely defined operators, which are essential for the development of a robust functional calculus in infinite dimensions.

4.2 Densely Defined Operators

Let $\{P_n\}_{n\geq 1}$ be a complete family of disjoint projections in a Hilbert space $\mathcal{H}$, that is, $P_m P_n = \delta_{mn} P_n$ and for all x in $\mathcal{H}$,

$$\sum_{n\geq 1} P_n x = x \tag{4.2.1}$$

Let $\{\epsilon_n\}_n \subseteq \mathbb{C}$, and set

$$D(T) = \left\{ x \in \mathcal{H} : \sum_{n\geq 1} \varepsilon_n P_n x \text{ converges} \right\}.$$

Claim : D is a dense linear subspace of $\mathcal{H}$.

Proof. Let

$$G = \bigcup_{N\geq 1} Ran\left(\sum_{n=1}^{N} P_n\right) = \left\{ x \in \mathcal{H} : \exists N \text{ s.t. } \sum_{n=1}^{N} P_n x = x \right\}.$$

To prove our claim, it suffices to show that G is a subset of $D(T)$, $G \subseteq D(T)$, and that G is dense in $\mathcal{H}$.

- $G \subseteq D(T)$. Let $x \in G$, then $\sum\limits_{j=1}^{J} P_j x = x$ for some J. Hence for all $N \geq J$,

$$\sum_{n=1}^{N} \varepsilon_n P_n x = \sum_{n=1}^{N} \varepsilon_n P_n \sum_{j=1}^{J} P_j x = \sum_{j=1}^{J} \varepsilon_j P_j x.$$

Thus, $\lim_{N\to\infty}\sum_{n=1}^{N}\varepsilon_n P_n x = \sum_{j=1}^{J}\varepsilon_j P_j x$ exists. So, $x \in D(T)$, that is, $G \subseteq D(T)$.

- G is dense in $\mathcal{H}$. Indeed, for all $x \in \mathcal{H}$, define the sequence $\{x_k\}_k$

$$x_k = \sum_{j=1}^{k} P_j x, \quad k \geq 1. \tag{4.2.2}$$

We have,

$$\left(\sum_{n=1}^{k} P_n\right) x_k = \sum_{n,j=1}^{k} \underbrace{P_n P_j}_{\delta_{nj} P_j} x = \sum_{j=1}^{k} P_j x = x_k,$$

so, $x_k \in G$. Furthermore,

$$\lim_{k\to\infty} x_k = \lim_{k\to\infty}\sum_{j=1}^{k} P_j x = \sum_{j\geq 1} P_j x = x \quad \text{by equation (4.2.1)}.$$

Hence, G is dense in $\mathcal{H}$.

□

Define the subspaces

$$E_n = Ran(P_n). \tag{4.2.3}$$

P_n being a projection, we have for all $x \in \mathcal{H}$, $x \in E_n \Leftrightarrow P_n x = x$. The family $\{P_n\}_n$ is disjoint and satisfies the equation (4.2.1). That yields

$$\mathcal{H} = \bigoplus_{n\geq 1} E_n.$$

Let's define on $D(T)$, the operator $T : D(T) \to Ran(T)$, given by

$$Tx := \sum_{n\geq 1} \varepsilon_n P_n x \tag{4.2.4}$$

The operator T is called a densely defined operator.

4.2.1 Proposition. The following statements are true:

1. For all $j \in \mathbb{N}$ ε_j are eigenvalues of T.
2. $\|T\| \geq \sup_j |\varepsilon_j|$.
3. If the rank of all P_n is finite and $\sum_n |\varepsilon_n|^2 < \infty$, then T is compact.

Proof. 1. Let $j \in \mathbb{N}$, show that ε_j is eigenvalue of T is equivalent to find a non-zero vector x such that $Tx = \varepsilon_j x$.

Claim: $E_j \neq \{0\}$ (otherwise we will have $P_j = 0$). Thus, let $0 \neq x \in E_j$, we have:

$$P_j x = x \text{ and } P_k x = 0 \quad \text{for all } k \neq j.$$

That give us,

$$Tx = \sum_{n\geq 1} \varepsilon_n P_n x = \sum_{n\geq 1, n\neq j} \varepsilon_n \underbrace{P_n x}_{0} + \varepsilon_j \underbrace{P_j x}_{x} = \varepsilon_j x.$$

Therefore, $\forall j$, ε_j is an eigenvalue of T with eigenspace associated E_j.

2. Let us first remark that, $\forall j, E_j \subseteq D(T)$ which implies that the supremum on $D(T)$ is greater than the supremum on E_j. Now using the definition of $\|T\|$ we have

$$\|T\| = \sup_{x\in D(T); x\neq 0} \frac{\|Tx\|}{\|x\|} \geq \sup_{x\in E_j; x\neq 0} \frac{\|T(x)\|}{\|x\|} \quad \forall j,$$

and that yields

$$\|T\| \geq \sup_{x\in E_j; x\neq 0} \frac{\|\varepsilon_j x\|}{\|x\|} = |\varepsilon_j| \quad \forall j.$$

That is,

$$\|T\| \geq \sup_j |\varepsilon_j|.$$

3. Suppose the rank of all P_n is finite and $\sum\limits_n |\varepsilon_n|^2 < \infty$, let's show that T is compact. **Approximation by Finite-Rank Operators** is one of the general approaches for demonstrating compactness. It consists of showing that the operator can be approximated by a sequence of finite-rank operators. A finite-rank operator is one that has a finite-dimensional range. If we can construct a sequence of finite-rank operators that converges to the given operator in some appropriate norm or topology, then we can establish its compactness.

Let's construct a sequence of operators of finite rank that converges to T (in operator norm).

Set

$$T_N = \sum_{n=1}^{N} \varepsilon_n P_n, \quad N \geq 1.$$

$\{T_N\}_N$ is a sequence of linear operators of finite rank (since the P_n are of finite rank). Let's show that T_N tends to T when N approaches ∞.

By definition, of the operator norm, we have,

$$\|T - T_N\| = \sup_{x\in D(T); \|x\|=1} \|\left(T - T_N\right)x\| \tag{4.2.5}$$

$\forall x \in D(T)$,

$$\|\left(T - T_N\right)x\|^2 = \|\sum_{n=N+1}^{\infty} \varepsilon_n P_n x\|^2 \leq \left(\sum_{n=N+1} |\varepsilon_n| \|P_n x\|\right)^2 \leq \sum_{n=N+1}^{\infty} |\varepsilon_n|^2 \sum_{n=N+1}^{\infty} \|P_n x\|^2$$

Since, for all $x \in \mathcal{H}$, $\sum\limits_{n\geq 1} P_n x = x$ that implies $\|x\|^2 = \|\sum_{n\geq 1} P_n x\|^2$. We then have

$$\|\left(T - T_N\right)x\| \leq R_N \|x\|^2,$$

where $R_N = \sum\limits_{n=N+1}^{\infty} |\varepsilon_n|^2$ is the rest of order N of the series $\sum\limits_{N\geq 1} |\varepsilon_N|^2$.

Equationn (4.2.5) becomes

$$\|T - T_N\| \leq \sqrt{R_N}. \tag{4.2.6}$$

Since $\sum_{N\geq 1} |\varepsilon_N|^2 < \infty$, $R_N \longrightarrow 0, N \longrightarrow \infty$. Hence,

$$\lim_{N\to\infty} T_N = T. \tag{4.2.7}$$

Therefore, T is compact.

□

4.2.2 Remark. Note, if $\sup_j |\varepsilon_j| = \infty$ then T is unbounded and so it can not be defined on all of $\mathcal{H}$ (c.f. Hellinger-Toepeiz Theorem [1]).

4.3 Construction of a Functional Calculus

Let $f : \{\varepsilon_n\} \longrightarrow \mathbb{C}$ be a function. Define the subset $D(f(T))$ of $\mathcal{H}$ by,

$$D(f(T)) = \left\{ x \in \mathcal{H} : \sum_{n\geq 1} f(\varepsilon_n) P_n x \text{ converges} \right\}.$$

One defines on $D(f(T))$ the operator $f(T) : D(f(T)) \to Ran(f(T))$ given by

$$f(T)x := \sum_{n\geq 1} f(\varepsilon_n) P_n x. \tag{4.3.1}$$

Note that $f(T)$ is densely defined by the claim in Section 4.2. Set

$$H_{\varepsilon_n} = \{f : \{\varepsilon_n\} \longrightarrow \mathbb{C}\} \quad \text{and} \quad D(H_{\varepsilon_n}T) = \left\{ x \in \mathcal{H} : \forall f \in H_{\varepsilon_n}, \sum_{n\geq 1} f(\varepsilon_n) P_n x \text{ converges} \right\}.$$

Now, the mapping $\Phi : H_{\varepsilon_n} \to \mathcal{L}(D(H_{\varepsilon_n}T))$ given by

$$\Phi(f)x := f(T)x := \sum_{n\geq 1} f(\varepsilon_n) P_n x. \tag{4.3.2}$$

is indeed a **Functional Calculus**.

4.3.1 Theorem. *Let T, defined in (4.2.4), the mapping Φ, given by (4.3.2), is a homomorphism of unital algebras.*

Proof. Let's prove that Φ is a homomorphism. Let $f, g \in H_{\varepsilon_n}, x \in D(H_{\varepsilon_n}T)$, show that

- $\Phi(f+g) = \Phi(f) + \Phi(g)$, that is $(f+g)T = f(T) + g(T)$, i.e. $(f+g)Tx = f(T)x + g(T)x$. By definition, one has:

$$(f+g)Tx = \sum_{n\geq 1} (f+g)(\varepsilon_n) P_n x = \sum_{n\geq 1} f(\varepsilon_n) P_n x + g(\varepsilon_n) P_n x.$$

Since the series $\sum_{n\geq 1} f(\varepsilon_n)P_n x$ and $\sum_{n\geq 1} g(\varepsilon_n)P_n x$ are convergent, we have:

$$(f+g)\,Tx = \sum_{n\geq 1} f(\varepsilon_n)P_n x + \sum_{n\geq 1} g(\varepsilon_n)P_n x = f(T)x + g(T)x. \tag{4.3.3}$$

- $\Phi\,(fg) = \Phi(f)\Phi(g)$, i.e. $(fg)\,Tx = f(T)g(T)x$.

$$\begin{aligned} f(T)g(T)x &= f(T)\left(\sum_{n\geq 1} g(\varepsilon_n)P_n x\right) \\ &= \sum_{m\geq 1} f(\varepsilon_m)P_m \left(\sum_{n\geq 1} g(\varepsilon_n)P_n x\right) \\ &= \sum_{m\geq 1}\sum_{n\geq 1} f(\varepsilon_m)g(\varepsilon_n)\delta_{m,n}P_n x \\ &= \sum_{n\geq 1} (fg)\,(\varepsilon_n)P_n x \\ &= (fg)\,Tx. \end{aligned}$$

- $\Phi(\mathbf{1}_{H_{\varepsilon_n}}) = I_d$, i.e., $\mathbf{1}_{H_{\varepsilon_n}}(T)x = x$.

$$\mathbf{1}_{H_{\varepsilon_n}}(T)x = \sum_{n\geq 1} \mathbf{1}_{H_{\varepsilon_n}}(\varepsilon_n)P_n x = \sum_{n\geq 1} P_n x = x. \tag{4.3.4}$$

□

For all $\lambda \notin \{\varepsilon_n\}$, define the subset of $\mathcal{H}$

$$D(R(\lambda)) = \left\{ x \in \mathcal{H} : \sum_{n\geq 1} (\varepsilon_n - \lambda)^{-1} P_n x \text{ converges} \right\}$$

One defines on $D(R(\lambda))$ the operator $R(\lambda) : D(R(\lambda)) \to Ran(R(\lambda))$ given by

$$R(\lambda)x := \sum_{n\geq 1} (\varepsilon_n - \lambda)^{-1} P_n x. \tag{4.3.5}$$

Note that $R(\lambda)$ is densely defined.

4.3.2 Proposition.

1. $R(\lambda) \subseteq D(T)$ and $(T-\lambda)R(\lambda) = \mathbf{1}$ on $D(R(\lambda))$.
2. $Ran(T) \subseteq D(R(\lambda))$ and $R(\lambda)(T-\lambda) = \mathbf{1}$ on $D(T)$.

Proof. 1. Let $x \in D(R(\lambda))$

$$\begin{aligned}
(T-\lambda)R(\lambda)x &= (T-\lambda)\sum_{n\geq 1}(\varepsilon_n-\lambda)^{-1}P_n x \\
&= T\sum_{n\geq 1}(\varepsilon_n-\lambda)^{-1}P_n x - \lambda\sum_{n\geq 1}(\varepsilon_n-\lambda)^{-1}P_n x \\
&= \sum_{m\geq 1}\varepsilon_m P_m\left(\sum_{n\geq 1}(\varepsilon_n-\lambda)^{-1}P_n x\right) - \lambda\sum_{n\geq 1}(\varepsilon_n-\lambda)^{-1}P_n x \\
&= \sum_{n\geq 1}\varepsilon_n(\varepsilon_n-\lambda)^{-1}P_n x - \sum_{n\geq 1}\lambda(\varepsilon_n-\lambda)^{-1}P_n x \\
&= \sum_{n\geq 1}(\varepsilon_n-\lambda)(\varepsilon_n-\lambda)^{-1}P_n x \\
&= \sum_{n\geq 1}P_n x \\
&= x.
\end{aligned}$$

2. Let $x \in D(T)$,

$$\begin{aligned}
R(\lambda)(T-\lambda)x &= R(\lambda)(Tx-\lambda x) \\
&= R(\lambda)\left(\sum_{n\geq 1}\varepsilon_n P_n x - \lambda x\right) \\
&= \sum_{m\geq 1}(\varepsilon_m-\lambda)^{-1}P_m\sum_{n\geq 1}\varepsilon_n P_n x - \lambda\sum_{n\geq 1}(\varepsilon_n-\lambda)^{-1}P_n x \\
&= \sum_{n\geq}\varepsilon_n(\varepsilon_n-\lambda)^{-1}P_n x - \lambda\sum_{n\geq 1}(\varepsilon_n-\lambda)^{-1}P_n x \\
&= \sum_{n\geq 1}(\varepsilon_n-\lambda)(\varepsilon_n-\lambda)^{-1}P_n x \\
&= x.
\end{aligned}$$

□

From the above proposition, we have that $R(\lambda) = (T-\lambda)^{-1}$.

4.3.3 Special case when $P_n = |\psi_n\rangle\langle\varphi_n|$. Let $\{\psi_n\}_n \subseteq \mathcal{H}$ be a Riesz basis of $\mathcal{H}$, meaning that

$$\psi_n = Ae_n.$$

where $\{e_n\}$ is an orthogonal basis of $\mathcal{H}$ and A is a bounded bijective operator. Then there exists a unique other Riesz basis $\{\varphi_n\}_n$ of $\mathcal{H}$ (see [8, Theorem 3.3.2, p. 59]) such that

$$x = \sum_{n=1}^{\infty}\langle\varphi_n, x\rangle\psi_n \quad \forall x \in \mathcal{H} \tag{4.3.6}$$

For all n, $\varphi_n = Be'_n$ where $\{e'_n\}$ is an orthogonal basis of $\mathcal{H}$ and B, a bounded bijective linear operator. The pair $\{(\psi_n, \varphi_n)\}_{n\in\mathbb{N}}$ is called a **Riesz biorthonormal basis** of $\mathcal{H}$ (i.e $\langle\psi_k, \varphi_l\rangle = \delta_{kl}$).

For $\varepsilon_n \in \mathbb{C}$, we define the operator T by

$$Tx = \sum_{n\geq 1} \varepsilon_n |\psi_n\rangle\langle\varphi_n| x \qquad (P_n = |\psi_n\rangle\langle\varphi_n|) \tag{4.3.7}$$

on

$$D(T) = \left\{ x \in \mathcal{H} : \sum_{n\geq 1} \varepsilon_n |\psi_n\rangle\langle\varphi_n, x\rangle \text{converges} \right\}.$$

4.3.4 Theorem. *If $\{\varepsilon_n\} \in l^2$ then T is bounded and*

$$\|T\| \leq c \left(\sum_{n\geq 1} |\varepsilon_n|^2 \right)^{1/2} \qquad \textit{for some } c < \infty. \tag{4.3.8}$$

Proof. Let $x \in D(T)$,

$$\begin{aligned}
\|Tx\|^2 &= \langle Tx, Tx\rangle \\
&= |\langle Tx, Tx\rangle| \\
&= \left| \sum_{m,n} \bar{\varepsilon}_m \varepsilon_n \Big\langle |\psi_m\rangle\langle\varphi|x, |\psi_n\rangle\langle\varphi_n|x \Big\rangle \right| \\
&= \left| \sum_{m,n} \bar{\varepsilon}_m \varepsilon_n \Big\langle A|e_m\rangle\langle e'_m|B^*x, A|e_n\rangle\langle e'_n|B^*x \Big\rangle \right| \\
&= \left| \sum_{m,n} \bar{\varepsilon}_m \varepsilon_n \langle Ae_m, Ae_n\rangle\langle e'_m, B^*x\rangle \left\langle e'_n, B^*x\right\rangle \right| \\
&\leq \sum_{m,n} |\bar{\varepsilon}_m \varepsilon_n| \underbrace{|\langle Ae_m, Ae_n\rangle|}_{\leq \|A\|^2} |\langle e'_m, B^*x\rangle| \left| \left\langle e'_n, B^*x\right\rangle \right| \\
&\leq \|A\|^2 \sum_{m,n} |\bar{\varepsilon}_m||\varepsilon_n||\langle e'_m, B^*x\rangle| \left| \left\langle e'_n, B^*x\right\rangle \right| \\
&\leq \|A\|^2 \left(\sum_n |\varepsilon_n| \left| \left\langle e'_n, B^*x\right\rangle \right| \right)^2 \\
&\leq \|A\|^2 \left(\sum_n |\varepsilon_n|^2 \right) \left(\sum_n |\langle e'_n, B^*x\rangle|^2 \right).
\end{aligned}$$

Since $\{e'_n\}$ is an orthogonal basis, we have

$$\sum_n |\langle e'_n, B^*x\rangle|^2 = \|B^*x\|^2.$$

Hence,

$$\|Tx\|^2 \leq \|A\|^2\|B\|^2 \left(\sum_n |\varepsilon_n|^2 \right) \|x\|^2.$$

Therefore,

$$\|T\|^2 \leq \|A\|\|B\| \left(\sum_n |\varepsilon_n|^2 \right)^{1/2} \quad \text{with } \|A\|\|B\| < \infty. \tag{4.3.9}$$

□

4.3.5 Proposition. $\forall x \in G, \quad \lambda \longmapsto R(\lambda)x$ is analytic on $\mathbb{C}\backslash\{e_n\}_{n\geq 1}$.

The proof follows from Theorem 2.3.13 of Section 2.

Let $\gamma = \bigcup_{j\geq 1} \gamma_j$, where γ_j is a circle centered at ε_j and only containing one point of $\{\varepsilon_n\}$. Suppose f is holomorphic in the interior of (piecewise holomorphic on the inside of the γ_n). As $\lambda \longmapsto f(\lambda)R(\lambda)x$ is continuous on each γ_n, the integral $\int_{\gamma_j} f(\lambda)R(\lambda)xd\lambda$ exists for all $x \in G$. Now for all $x \in G$, there exists N such that $x = \sum_{n=1}^{N} P_n x$ and $R(\lambda)x = \sum_{n=1}^{N} (\varepsilon_n - \lambda)^{-1} P_n x$ and so

$$\int_{\gamma_j} f(\lambda)R(\lambda)x\, d\lambda = \sum_{n=1}^{N} \left(\int_{\gamma_j} \frac{f(\lambda)}{\varepsilon_n - \lambda} d\lambda \right) P_n x = \begin{cases} 0 & \text{if } j > N \\ -2\pi i f(\varepsilon_j)x & \text{if } j \leq N. \end{cases} \tag{4.3.10}$$

4.4 Vector Valued-Functions

Let X be a complex linear space, $I \subseteq \mathbb{R}$, an interval of R, a **vector valued-function** on X, is a single-valued mapping $x(t)$ form I into X.

The point $y \in X$ is called the limit of the vector-valued function $x(t)$ as t tends to a, denoted by $y = \lim_{t\to a} x(t)$, if

$$\langle f, y\rangle = \lim_{t\to a} \langle f, x(t)\rangle$$

is satisfied for all $f \in X^*$

4.4.1 Riemann Intergral. [6] Let $x(t)$ be a continuous vector-valued function on the finite interval $[a, b]$. The integral of the vector $x(t)$ over the interval $[a, b]$ is defined as follows: let f be a linear form on X, $f \in X^*$, then the scalar product $\langle f, x(t)\rangle$ is continuous for all t in $[a, b]$. Thus, the integral of $\langle f, x(t)\rangle$ over $[a, b]$ exists in the sense of **Riemann**. Hence, $\exists!\ y \in X$ (Riesz representation therorem) s.t.

$$\int_a^b \langle f, x(t)\rangle dt = \langle y, f\rangle, \quad f \in X^*.$$

We write

$$y = \int_a^b x(t)dt. \tag{4.4.1}$$

Let $\{e_1, e_2, \cdots, e_N\}$ a basis of X, suppose $x(t) = \sum_{i=1}^{N} x_i(t)e_i$, then the integral in 4.4.1 can be written as

$$\int_a^b x(t)dt = \sum_{i=1}^{N} \int_a^b x_i(t)dte_i. \tag{4.4.2}$$

Thereby, a continuous vector-valued function is integrated by means of its coordinates, with respect to any basis. The integral satisfies the following properties:

- $\int (\alpha x(t) + \beta y(t))dt = \alpha \int x(t) + \beta \int y(t)dt.$
- $\| \int_a^b x(t)\| \leq \int_a^b \|x(t)\|dt$

4.4.2 Vector valued Function of complex argument. Let the map

$$x : G \subseteq \mathbb{C} \to X, \quad z \longmapsto x(z) \tag{4.4.3}$$

be a continuous vector-valued function in G, τ, a piecewise smooth bounded contour in G, then the Riemann integral $\int_\tau x(z)dz$ is defined by

$$\langle f, \int_\tau x(z)dz\rangle = \int_\tau \langle f, x(z)\rangle dz, \quad f \in X^*.$$

For a holomorphic function $x(z)$ in a simply connected region τ, the Cauchy's integral theorem

$$\int_\tau x(z)dz = 0$$

holds.

Proof. [19, Corollary 5.13, p. 201] Let's first recall a corollary from the Hahn-Banach theorem,

4.4.3 Corollary. Let X be a normed vector space and let $x \in X$. If $\langle f, x\rangle = 0$, for all continuous linear functional f acting on X, then $x = 0$.

We can initially observe that for every bounded linear functional f on X, the inner product $\langle f, x(z)\rangle$ is holomorphic on τ. By applying the Cauchy theorem, we have $\langle f, \int_\tau x(z)dz\rangle = \int_\tau \langle f, x(z)\rangle dz = 0$. It then follows by the corollary 4.4.3 that $\int_\tau x(z)\,dz = 0$. □

4.5 Operator-Valued Functions

4.5.1 Operator valued functions of a scalar argument. Let X be a complex linear space, the map

$$A : I \subseteq \mathbb{R} \to \mathcal{L}(X), \quad t \longmapsto A(t) \tag{4.5.1}$$

is called an **operator-valued function**.

$B \in \mathcal{L}(X)$ is called the limit of the operator-valued function $A(t)$ as t tends to a, $B = \lim_{t\to a} A(t)$, if

$$\lim_{t\to a} A(t)x = Bx$$

holds for all $x \in X$. That is equivalent to

$$\lim_{t\to a}\langle A(t)x, f\rangle = \langle Bx, f\rangle \quad \forall x \in X, f \in X^*$$

The existence and uniqueness of $\int_a^b A(t)dt$ follows from the following proposition and from the Riesz representation theorem.

4.5.2 Riemann Integral. Let $A(t)$ be a continuous operator-valued functions on the finite interval $[a, b]$. The integral of the operator $A(t)$ over the interval $[a, b]$ is defined by:

$$\left\langle \int_a^b A(t)dtx, f \right\rangle = \int_a^b \langle A(t)x, f\rangle dt \quad \forall x \in X,\ f \in X^*. \tag{4.5.2}$$

4.5.3 Proposition. The integral $\int_a^b \langle A(t)x, f\rangle\, dt$ is a bounded sesquilinear functional in x and f.

Proof. Let's establish the bounded sesquilinear properties for this integral.

- **Linearity in the first argument:** For all x, y in X, $f \in X^*$ and α, β complex scalars, we have:
$$\int_a^b \langle A(t)\left(\alpha x + \beta y\right), f\rangle\ dt = \int_a^b \langle \alpha A(t)x + \beta A(t)y, f\rangle\, dt = \alpha \int_a^b \langle A(t)x, f\rangle\ dt + \beta \int_a^b \langle A(t)y, f\rangle\ dt$$
- **Antilinearity in the second argument:** For all x in X, $f, g \in X^*$ and α, β in $\mathbb{C}$, we have:
$$\int_a^b \langle A(t)x, (\alpha f + \beta g)\rangle\ dt = \bar{\alpha} \int_a^b \langle A(t)x, f\rangle\ dt + \bar{\beta} \int_a^b \langle A(t)y, f\rangle\ dt$$
- **Boundedness:** For all $x \in X, f \in X^*$, we have:
$$\left| \int_a^b \langle A(t)x, f\rangle \right| \leq \int_a^b |\langle A(t)x, f\rangle| \leq (b-a) \sup_{t\in[a,b]} |A(t)x| \|f\| \leq C(b-a)\|x\|\|f\|. \tag{4.5.3}$$

□

Then by the Riesz Representation Theorem [2], there exists a unique $B \in \mathcal{L}(X)$ such that

$$\int_a^b \langle A(t)x, f\rangle dt = \langle Bx, f\rangle, \quad x \in X, f \in X^*, \tag{4.5.4}$$

and the operator B is given by

$$B = \int_a^b A(t)\, dt. \tag{4.5.5}$$

4.5.4 Operator valued Function of complex argument. Let the map

$$A : G \subseteq \mathbb{C} \to \mathcal{L}(X), \quad z \longmapsto A(z) \tag{4.5.6}$$

be a continuous operator-valued functions in G, τ, a piecewise smooth bounded contour in G, then the Riemann integral $\int_\tau A(z)dz$ is defined by

$$\left\langle \int_\tau A(z)x, f dz \right\rangle = \int_\tau \langle A(z)x, f\rangle\, dz, \quad x \in X,\ f \in X^*.$$

For a holomorphic operator $A(z)$ in a simply connected region τ, the Cauchy's integral theorem

$$\int_\tau A(z)dz = 0$$

is valid.

Follow the same approach as in the Section 4.4.2 in the case of vector-valued function, for the proof.

5. Conclusion

This work has undertaken a comprehensive exploration of the functional calculus for linear operators, with a particular emphasis on providing a thorough understanding of the Riesz functional calculus. By systematically studying the properties of linear operators, their spectra, and the resolvent operators, a robust theoretical foundation has been established. Building upon this foundation, the investigation has delved into the development of various functional calculi, including the polynomial calculus and the power series calculus, culminating in the construction of the Riesz functional calculus. A key aspect of this investigation has been the detailed analysis of the Spectral Mapping Theorem, which has provided crucial insights into the interplay between operator-valued functions and the underlying spectral structure.

The culmination of this work has been the construction of a functional calculus for the specialized class of densely defined operators on normed vector spaces. This approach involves the characterization of a densely defined operator by a complete family of disjoint projections in a Hilbert space, along with a sequence of complex numbers. This specialized framework has enabled the development of a functional calculus that can be effectively applied to this class of operators.

The insights gained through this focused investigation have broader implications, contributing to the understanding of operator theory and opening new avenues for research and applications. One intriguing open question that remains is how to construct the functional calculus for unbounded operators (but with discrete spectrum).

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